SOLUTIONS.

330

FIND the sum of the series

$$1^2 + 3^2 + 6^2 + 10^2 + 15^2 + \ldots + [\tfrac{1}{2} n (n+1)]^2.$$

[*Artemas Martin.*]

SOLUTION.

The fifth differences vanish and the first terms of the difference series are 1, 8, 19, 18, 6, 0; hence

$$S = {}_nC_1 + {}_nC_2 \,.\, 8 + {}_nC_3 \,.\, 19 + {}_nC_4 \,.\, 18 + {}_nC_5 \,.\, 6\,,$$

where ${}_nC_1$, etc. are the binomial coefficients. [*Geo. R. Dean.*]

332

FOUR equianharmonic points give four triangles which have four circumcircles. Show that the inverses of any point with regard to these four circles are equianharmonic. [*Frank Morley.*]

SOLUTION.

It is proved in Salmon's Conic Sections, § 54, that if the perpendiculars from three points a_1, a_2, a_3 on the sides of a triangle $c_1c_2c_3$ meet at a point a_4, then also the perpendiculars from c_1, c_2, c_3 on the sides of $a_1a_2a_3$ meet at a point c_4. And this can be readily proved by elementary geometry, by observing that the condition of either concurrence is, that in the hexagon a_1, c_3, a_2, c_1, a_3, c_2 the sum of the squares of three alternate sides is equal to the sum of the squares of the other three sides.

Now

$$\left|\frac{c_2 - c_3}{c_1 - c_3}\right| = \frac{\sin c_2c_1c_3}{\sin c_3c_2c_1} = \frac{\sin a_2a_4a_3}{\sin a_3a_4a_1},$$

$$\left|\frac{c_1 - c_4}{c_2 - c_4}\right| = \frac{\sin c_4c_2c_1}{\sin c_4c_1c_2} = \frac{\sin a_1a_3a_4}{\sin a_4a_3a_2};$$

whence

$$\left|\frac{c_2 - c_3 \,.\, c_1 - c_4}{c_1 - c_3 \,.\, c_2 - c_4}\right| = \frac{\sin a_2a_4a_3 \,.\, \sin a_1a_3a_4}{\sin a_4a_3a_2 \,.\, \sin a_3a_4a_1} = \left|\frac{a_2 - a_3 \,.\, a_1 - a_4}{a_2 - a_4 \,.\, a_1 - a_3}\right|.$$

Hence two corresponding cross-ratios of the two tetrads of points have equal

absolute values. They have also congruent amplitudes; for

$$\text{am}\,(c_2 - c_3) \equiv \text{am}\,(a_1 - a_4) - \pi/2\,,$$

$$\text{am}\,(c_1 - c_4) \equiv \text{am}\,(a_2 - a_3) + \pi/2\,;$$

and, therefore,

$$\text{am}\,(c_2 - c_3 \,.\, c_1 - c_4) \equiv \text{am}\,(a_2 - a_3 \,.\, a_1 - a_4)\,.$$

Hence corresponding cross-ratios are equal.

In particular, let c_1, c_2, c_3 be the centres of the circles $a_2a_3a_4$, $a_3a_4a_1$, $a_4a_1a_2$; then c_4 is the centre of the circle $a_1a_2a_3$; and the theorem is: The inverses of the point ∞, with regard to the four circles through three of any four points a, have the same cross-ratios as the four points. This is a covariantive statement, in which we can substitute any other point for the point ∞.

[*Frank Morley.*]

339

Show that the areas of the curves

$$y = px^2 + qx + r\,, \text{ and } y = mx^3 + px^2 + qx + r\,,$$

taken between the limits $x + h$ and $x - h$, are given by the formulæ

$$\Omega = 2yh + \tfrac{2}{3}ph^3\,, \text{ and } \Omega = 2yh + (\tfrac{2}{3}p + 2mx)h^3\,,$$

respectively. [*W. H. Echols.*]

SOLUTION I.

From the well-known formula

$$\Omega = \int y dx$$

we have for the first curve

$$\Omega = \int_{x-h}^{x+h} (px^2 + qx + r)\,dx = \tfrac{1}{3}px^3 + \tfrac{1}{2}qx^2 + rx\Big]_{x-h}^{x+h}$$

$$= 2h\,(px^2 + qx + r) + \tfrac{2}{3}ph^3 = 2yh + \tfrac{2}{3}ph^3\,;$$

and for the second curve

$$\Omega = \int_{x-h}^{x+h} (mx^3 + px^2 + qx + r)\,dx = \tfrac{1}{4}mx^4 + \tfrac{1}{3}px^3 + \tfrac{1}{2}qx^2 + rx\Big]_{x-h}^{x+h}$$

$$= 2\text{h}\,(mx^3 + px^2 + qx + r) + (\tfrac{2}{3}p + 2mx)\,h^3 = 2yh + (\tfrac{2}{3}p + 2mx)\,h^3\,.$$

[*Artemas Martin.*]

SOLUTION II.

By the familiar formula called Simpson's rule the mean value of

$$y = mx^3 + px^2 + qx + r$$

is the sixth part of the sum of the extreme values plus four times the middle value. But

$$(x + h)^3 + (x - h)^3 + 4x^3 = 6x^3 + 6h^2x\,,$$

$$(x + h)^2 + (x - h)^2 + 4x^2 = 6x^2 + 2h^2\,,$$

$$(x + h) + (x - h) + 4x = 6x\,;$$

$$y_m = mx^3 + px^2 + qx + r + mh^2x + \tfrac{1}{3}ph^2$$

$$= y + h^2(mx + \tfrac{1}{3}p)\,,$$

and

$$A = 2hy + h^3(2mx + \tfrac{2}{3}p)\,.$$

Putting $m = 0$, we have the other result. [*W. M. Thornton.*]

SOLUTION III.

As applications of

$$\tfrac{1}{2}\int_{x-h}^{x+h} fx\,.\,dx = hfx + \frac{h^3}{3!}f''x + \frac{h^5}{5!}f^{\text{iv}}x + \ldots\,,$$

we have for the first one,

$$\tfrac{1}{2}\,\Omega = hy + \frac{h^3}{3!}2p\,;$$

for the second

$$\tfrac{1}{2}\,\Omega = hy + \frac{h^3}{3!}(3\,.\,2mx + 2p)\,.$$

[*W. H. Echols.*]

346

A WOODEN hemisphere floats in water, vertex down, with $1/n$ of its axis immersed. Find the specific gravity of the hemisphere.

[*Artemas Martin.*]

SOLUTION.

Let s be the specific gravity of the hemisphere and r its radius. The weight of the hemisphere equals the weight of the water displaced, or,

$$\tfrac{2}{3}\pi r^3 s = \pi\left[\frac{r}{n}\right]^2\left[r - \tfrac{1}{3}\,.\,\frac{r}{n}\right],$$

which gives

$$s = \frac{3n - 1}{2n^3}\,.$$

[*W. O. Whitescarver.*]

352

A SOLID sphere and a solid cylinder of equal radii roll from rest down the same inclined plane; compare the times of their descent.

[*Artemas Martin.*]

SOLUTION.

By the principle of vis viva,

$$mv^2 + I\omega^2 = 2mgs \sin i .$$

The motion being pure rolling,

$$v = r\omega .$$

For the sphere,

$$I = \tfrac{2}{5}\, mr^2 .$$

For the cylinder,

$$I = \tfrac{1}{2}\, mr^2 .$$

Substituting these values and reducing, we have, for the sphere,

$$v = \frac{ds}{dt} = \sqrt{\tfrac{5}{14}\, gs \sin i} ;$$

and for the cylinder,

$$v = \frac{ds}{dt} \cdot = \sqrt{\tfrac{1}{3}\, gs \sin i} .$$

Integrating, we get

$$\sqrt{s} = t \sqrt{\tfrac{5}{14}\, g \sin i} , \qquad \sqrt{s} = t' \sqrt{\tfrac{1}{3}\, g \sin i} ;$$

whence

$$\frac{t}{t'} = \sqrt{\tfrac{14}{15}} .$$

[*S. T. Moreland; M. C. Andrews; C. E. Mendenhall; G. R. Dean.*]

EXERCISE.

357

GIVEN $\tan_\kappa w = x + iy$, wherein

$$\tan_\kappa w = \varkappa \left(e^{w/\kappa} - e^{-w/\kappa}\right)/\left(e^{w/\kappa} + e^{-w/\kappa}\right) ,$$

$$w = u + iv , \quad i = \sqrt{-1} ,$$

and

$$\varkappa = m (\cos \beta + i \sin \beta) ;$$

determine u and v as real quantities in terms of x, y, m, and β.

[*Irving Stringham.*]